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A Non-Linear Analysis of Steady Surface Waves on a Thin Sheet of Viscous Liquid Flowing Down an Incline

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SUMMARY
A thin sheet of viscous liquid flows slowly down an uneven incline. The main conclusion derived in this paper is that the only effect of a variation in the bed from that of a uniform slope is noticed in the upstream direction; the flow has constant depth downstream from this variation.

1. Introduction

The flow in an open channel contains much of the theory of mathematical hydraulics. When the channel, such as a river, is inclined with an irregular bed, any steady flow results through a balance of the gravitational forces with those due to the pressure gradient and the viscous stress forces within the liquid. The presence of viscosity in the basic mathematical equations causes considerable difficulty, and one of the main techniques in previous theories is to replace all the viscous forces by a frictional resistance term derived from the empirical formula due to Manning. Stoker [1] devotes a section of his text "Water Waves" to such a theory; there it is applied to the large scale problem which considers the flow at the junction of two rivers.

A theory for the non-linear gravity waves in a sheet of viscous liquid, where the viscous terms have been fully incorporated through the Navier-Stokes equation, has recently been presented by Mei [2]. In this paper Mei considers the unsteady flow down an inclined plane, and observes different types of wave motion. He restricts his study to situations with low Reynolds and Froude numbers so that, with basically a Stokes flow, the inertia terms are neglected. A formal expansion procedure (due to Lin and Clark in their development of inviscid long wave theory) then enabled the basic non-linear partial differential equation for the wave amplitude to be formulated.

The only attempt known to the author at considering the steady motion down an incline of variable profile is that of P. Smith [3]. It is a linear analysis that belongs to the "infinitesimal wave" class. The exact solution of the Navier-Stokes equations, which represents the steady flow of constant depth down an inclined plane (see Berker [4]), was taken to be the basic flow; the perturbation scheme was developed with this as the zero order solution. A Fourier analysis allowed different bed profiles to be considered.

In this paper we essentially generalize the analysis of Mei to include variations in the profile of the stream bed. Although the equation is given for the unsteady case, our main interest is in the steady state problem. Here, the resultant non-linear, ordinary differential equation is of the first order only. Numerical integration for different profiles of the bed follow with little difficulty, though direct analytical results are possible for just one particular case. Since completing the analysis described in this paper I have become aware of the later work of P. Smith [5], a report of which appears elsewhere in this Journal. He provides an intuitive deduction of the same basic differential equation, and then solves this numerically for a sinusoidal bed profile.

2. Formulation

We consider the two-dimensional laminar flow down an incline whose mean slope has the gradient tan θ. The fluid is incompressible with a constant coefficient of viscosity. The x'-axis is set down the line of mean slope, and the y'-axis is normal to it so that the profile of the bottom can be expressed by $y' = h'(x')$. When the flow is steady, the resultant profile of the free surface is written $y' = \eta'(x')$. We further define u' and v' to be the velocities parallel to the x' and y' axes respectively at the point $P(x', y')$; the pressure is p'.

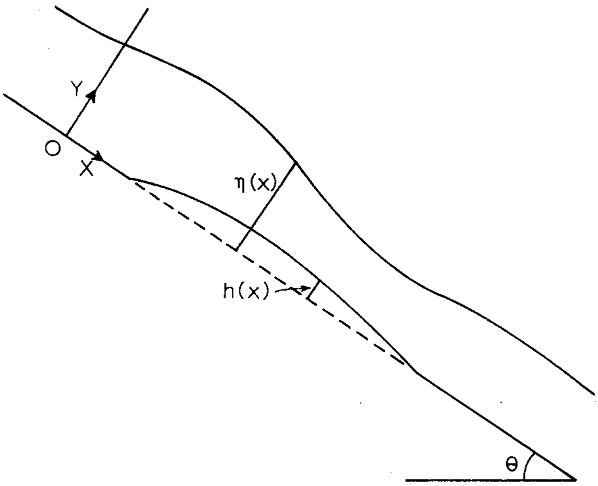

Figure 1

All these physical quantities are now made non-dimensional by the following transformations: $(x', y', \eta', h') = L(x, y, \eta, h)$, $(u', v') = U(u, v)$ and $p' = (\rho\nu U/L)p$, where ρ is the density, ν the kinematic viscosity and L, U are length and velocity scales.

The equations of motion are taken to be the Navier-Stokes equations

$$u_x + v_y = 0\,, \tag{1}$$

$$uu_x + vu_y = -\frac{1}{R}p_x + \frac{1}{F^2}\alpha + \frac{1}{R}(u_{xx} + u_{yy})\,, \tag{2}$$

$$uv_x + vv_y = -\frac{1}{R}p_y - \frac{1}{F^2}\beta + \frac{1}{R}(v_{xx} + v_{yy})\,, \tag{3}$$

where $\alpha = \sin\theta$, $\beta = \cos\theta$ and

$$R = \frac{UL}{\nu}\,, \quad F^2 = \frac{U^2}{gL}\,. \tag{4}$$

Clearly R is the Reynolds number and, with g the acceleration due to gravity, F^2 is the Froude number for the described motion. On the free surface profile $y = \eta(x)$ we have

$$\begin{aligned} &u\eta_x = v\,, \\ &(p - 2u_x)\eta_x + (u_y + v_x) = 0\,, \\ &(p - 2v_y) + (u_y + v_x)\eta_x = 0\,. \end{aligned} \tag{5a,b,c}$$

The condition (5a) is the kinematic condition, and the other two satisfy the requirement for the stress to be zero on $y = \eta$ when the surface is assumed to be free of external stress and the effect of surface tension. Also,

$$uh_x = v \quad \text{and} \quad u + vh_x = 0 \quad \text{on} \quad y = h(x) \tag{6a,b}$$

indicate the no-slip conditions.

When $h(x) \equiv 0$ there is the exact solution

$$u = \frac{R\alpha}{F^2}(dy - \tfrac{1}{2}y^2), \quad v = 0 \quad \text{and} \quad p = \frac{R\beta}{F^2}(d-y), \tag{7}$$

where d is a constant; see Berker [4]. The length $d' = Ld$, in fact, is the constant depth of the liquid, and is taken to be the representative length scale in the direction normal to the bed in all that follows.

3. The Basic Differential Equation

So far the formulation has been completely general; we now make the basic physical assumptions, following Mei. When the forces due to the viscous stresses and the pressure gradient are of equal importance, the Reynolds number is small; if, further, the effects of gravity are of the same order as these terms, then the Froude number is also small. Therefore, our basic restriction within the following analysis is to write

$$R = F^2 = \varepsilon, \tag{8}$$

and then set $\varepsilon \ll 1$. Consequently, the length scale $L = (U\nu/g)^{\frac{1}{2}}$, and, when we further take $U = (gd')^{\frac{1}{2}}$, the ratio $d = d'/L$ is equal to ε. This constitutes a "shallow liquid" approximation, where both h and η are $O(\varepsilon)$. Also, in order that the following analysis can in any way be tractable, we assume that all the derivatives of h and η are $O(\varepsilon)$.

The procedure now is to introduce the stream function $\psi(x, y)$ by

$$u = \psi_y \quad \text{and} \quad v = -\psi_x \tag{9}$$

to satisfy (1), then write

$$\psi = \sum_{n=0}^{\infty} y^n \psi_n(x), \qquad p = \sum_{n=0}^{\infty} y^n p_n(x). \tag{10}$$

It is noted that

$$\psi = \alpha(\tfrac{1}{2}\varepsilon y^2 - \tfrac{1}{6}y^3) \tag{11}$$

is the stream function corresponding to the velocities given in (7) when $R = F^2 = \varepsilon$.

Before the series (10) are substituted into the equations of motion (2) and (3), some understanding is required of the orders of magnitude of ψ_n and p_n for different n. It is first observed that the derivatives of ψ_n and p_n are of the same order in ε as (or at least no smaller than) the order of ψ_n and p_n respectively. Now, it is seen from (11) that ψ is $O(\varepsilon^3)$ when $h(x) \equiv 0$ and $\eta(x) \equiv \varepsilon$, but with only slow variations in the x-direction this must also be valid for all flows with variable η and h in a thin sheet of liquid. Hence, $\psi_0 = O(\varepsilon^3)$, $\psi_1 = O(\varepsilon^2)$, $\psi_2 = O(\varepsilon)$ and $\psi_3 = O(1)$. A similar consideration of the pressure as given in (7) shows that $p_0 = O(\varepsilon)$ and $p_1 = O(1)$. The orders of magnitude of the remaining quantities are found from the equations of motion.

The expansions (10) are substituted into the momentum equations; when we equate the coefficients of y there follows

$$-p_{0x} + \alpha + \psi_{1xx} + 6\psi_3 = \varepsilon(\psi_1\psi_{1x} - 2\psi_{0x}\psi_2), \tag{12a}$$

$$-p_{1x} + \psi_{2xx} + 24\psi_4 \quad = \varepsilon(2\psi_1\psi_{2x} - \psi_{0x}\psi_3), \tag{12b}$$

etc., from (2); also

$$p_1 + \beta + 2\psi_{2x} + \psi_{0xxx} = \varepsilon(\psi_{0xx}\psi_1 - \psi_0\psi_{1xx}), \tag{12c}$$

$$2p_2 + 6\psi_{3x} + \psi_{1xxx} \quad = \varepsilon(\psi_1\psi_{1xx} + 2\psi_{0xx}\psi_2 - \psi_{1x}^2 - 2\psi_{0x}\psi_{2x}), \tag{12d}$$

etc., from (3). In each of the above the right hand sides represent the complete inertia terms; they are much smaller than the other terms, and consequently are neglected. As the motion is a Stokes flow with $R \ll 1$, this was to be expected. The general pattern of the analysis can now be observed, for the infinite set of equations (12) enable us to solve for ψ_n ($n \geqq 3$) and p_n ($n \geqq 1$) in terms of the four functions ψ_0, ψ_1, ψ_2 and p_0. In particular, we have $\psi_3 = -\frac{1}{6}\alpha + O(\varepsilon)$ from (12a), and $p_1 = -\beta + O(\varepsilon)$ from (12c). Both ψ_{3x} and p_{1x} are therefore $O(\varepsilon)$; (12b,d) then show that ψ_4 and p_2 must be $O(\varepsilon)$. The process has now been set up, as we consider each of the equations (12) in turn it becomes clear that ψ_n ($n \geqq 4$) and p_n ($n \geqq 2$) are all $O(\varepsilon)$. When we substitute for ψ_n ($n \geqq 3$) and p_n ($n \geqq 1$) into the boundary conditions (5), (6) there result five equations from which ψ_0, ψ_1, ψ_2 and p_0 can be eliminated to give the final differential equation for η that we require.

This elimination is straightforward, and we do not dwell on it here; the details are left for an appendix. It is sufficient to note that

$$\begin{aligned} 6\psi_3 &= -\alpha + \beta\eta_x + O(\varepsilon^2), \quad 2\psi_2 = \alpha\eta - \beta\eta\eta_x + O(\varepsilon^3), \\ p_1 &= -\beta - 2\alpha\eta_x + O(\varepsilon^2), \quad p_0 = \beta\eta + O(\varepsilon^2), \\ \psi_1 &= -2h\psi_2 - 3h^2\psi_3 + O(\varepsilon^4), \quad \psi_{0x} = -h\psi_{1x} - h^2\psi_{2x} - h^3\psi_{3x} + O(\varepsilon^5) \end{aligned}$$

are substituted into the kinematic surface condition to give directly

$$\alpha\{(\eta-h)^3\}_x - \beta\{\eta_x(n-h)^3\}_x = O(\varepsilon^5). \tag{13}$$

After integration we have

$$\alpha(\eta-h)^3 - \beta\eta_x(\eta-h)^3 = \alpha\varepsilon^3 + O(\varepsilon^5)\ ; \tag{14}$$

the constant of integration follows from the case $h = 0$, where $\eta = \varepsilon$ and $\eta_x = 0$. Therefore the resultant non-linear differential equation for the free surface profile is of the first order only.

Several comments can be made at this juncture before solutions of the differential equation are considered. In (14) the first term is $O(\varepsilon^3)$, while the second is $O(\varepsilon^4)$; if this smaller term is neglected, then there follows the trivial approximation $\eta - h \simeq \varepsilon$. However, these terms can reasonably be considered in some degree of balance when the mean gradient of the incline $\tan\theta$ is taken to be small in some quite general manner.

Secondly, a physical interpretation is now given of the purely formal expansion procedure that has led to the equation (14). For the pressure only p_0 and p_1 were required; together these show $p' \simeq \rho g\beta(\eta' - y')$. This indicates that the balance of momentum in the y' direction is satisfied by the hydrostatic pressure. As a result, $p' \simeq 0$ on the free surface. Also, the above analysis shows $2\psi_2 + 6\eta\psi_3 \simeq 0$; this corresponds to $\partial u'/\partial y' \simeq 0$ on $y' = \eta'(x')$. These dominate the two dynamic conditions (5b,c) to be satisfied on the liquid surface. The above statements form the basic assumptions of P. Smith [5], consequently the resultant differential equation (14) (in this paper) and (4.2) in [5] are equivalent.

Further, the velocity u is given by

$$u(x, y) = \tfrac{1}{2}(y-h)(2\eta - y - h)(\alpha - \beta\eta_x) + O(\varepsilon^4).$$

The conservation of mass is observed, to the third order in ε, with the result

$$\int_h^\eta u\,dy = \tfrac{1}{3}\alpha\varepsilon^3.$$

So far the study has been restricted to steady flows; this has been for analytical simplicity. When the motion is unsteady the equation corresponding to (13) is found by amalgamating the calculations of Mei with those followed above. Here the result is just stated. If the physical time variable t' is written as Tt, where t is non-dimensional, and the Strouhal number L/UT is written as τ, we define $\tau = \varepsilon^2$. The partial differential equation is then

$$\varepsilon^2\eta_t + \alpha(\eta-h)^2(\eta-h)_x = \tfrac{1}{3}\beta\{\eta_x(\eta-h)^3\}_x + O(\varepsilon^5)\ ;$$

this is equivalent to that discovered by Mei when $h(x) \equiv 0$. As a further generalization, the equation for three-dimensional, time dependent, flow is given. We take Oz to be the axis perpendicular to the x, y-plane, and so lying on the face of the incline. The same formal analysis shows the differential equation for $\eta(x, z, t)$ to be

$$\varepsilon^2\eta_t+\alpha(\eta-h)^2(\eta-h)_x=\tfrac{1}{3}\beta\{\eta_x(\eta-h)^3\}_x+\tfrac{1}{3}\beta\{\eta_z(\eta-h)^3\}_z+O(\varepsilon^5)\,.$$

This reduces to the three dimensional equation given by P. Smith [5] when the flow is time dependent.

In the final comment we reformulate the equation when $h=O(\varepsilon^2)$. The resultant free surface elevation is written $\eta=\varepsilon+a$ with $a=O(\varepsilon^2)$; then $a(x)$ is given by

$$\beta\varepsilon^2 a_x-3\alpha\varepsilon(a-h)-3\alpha(a-h)^2=O(\varepsilon^5)\,.$$

In evaluating the error on the right-hand side it is necessary to repeat the analysis that resulted in (14), beginning from the initial equations and conditions (1)–(6). With α/β small, the last term on the left hand side can also be neglected to give a linear differential equation with the solution

$$a(x)\equiv\tau^{-1}e^{x/\tau}\int_x^{x_0}h(u)e^{-u/\tau}du \tag{15}$$

when $\tau=\beta\varepsilon/3\alpha$ and x_0 is a constant. Such a solution comes more under the heading of a "infinitesimal wave". However, (15) is less general than the linear analysis of P. Smith [3], who was not restricted to the "shallow liquid" approximation that we required for the basic differential equation to be derived. In fact the solution (15) provides the common ground between the two theories.

4. Analytical Solutions

It appears that solutions of the differential equation (14) by direct analysis are possible only when dh/dx is constant. The main features are revealed when we take dh/dx to be zero, which implies that $h(x)\equiv 0$, with no loss of generality. Now if the solution is to be valid for all values of x, it is immediately seen that we must have the constant depth $\eta(x)\equiv\varepsilon$. However, if the domain of x is restricted in some way, then the general solution of (14) can be written in the form

$$\frac{\alpha}{\beta}(x+k)=\eta+\tfrac{1}{6}\varepsilon\log\frac{(\eta-\varepsilon)^2}{\eta^2+\eta\varepsilon+\varepsilon^2}-\frac{1}{\sqrt{3}}\varepsilon\arctan\frac{2\eta+\varepsilon}{\sqrt{3}\varepsilon} \tag{16}$$

for some constant k. As η tends to ε this shows $\eta-\varepsilon\sim\pm\exp\{(3\alpha/\beta\varepsilon)(x+k)\}$. Therefore $\eta\to\varepsilon$ as $x\to-\infty$, and if $\eta=\eta_0$ at $x=0$, the end point of the domain $-\infty<x\leqq 0$, then the constant k can be evaluated. In fact $(\alpha/\beta)k$ is equal to the right hand side of (16) with the constant η_0 in place of η. However, if the domain considered in $0\leqq x<\infty$, and it is required that $\eta\to\varepsilon$ as $x\to\infty$, then the only possible solution is $\eta\equiv\varepsilon$ throughout this domain. Physically, these observations are sufficient to show that any effect of a change in the bed of the stream is felt upstream of that change, and if the slope is uniform downstream from a certain position then the depth of the stream is constant beyond that point.

Indirect means of gaining analytical results are always possible; (i.e. given $\eta(x)$, find the profile $h(x)$ that creates such a wave). The one noted here is that the long sinusoidal wave $\eta=\sigma\sin\lambda x$, $\sigma=O(\varepsilon)$ and $\lambda\ll 1$, is created by the profile $h=\sigma\sin\lambda(x-\tau)$. The distance $\tau=\beta\varepsilon/3\alpha$ represents the lag of the free surface behind that of the bed. P. Smith [5] has gained more general numerical solutions for this sinusoidal wave.

5. Numerical Solutions

The first order differential equation has been integrated numerically for different profiles $h(x)$; those presented in Fig. 2 display the dominant features present. As noted in the previous section, the effect of a change in the bed is felt well upstream, but not at all downstream of the variation. Also, just downstream of the highest point of the bed there is the lowest point of the stream; the flow will consequently have its maximum velocity close to such positions. When there is the deepest hollow in the bed this state is reversed. The linear analysis of P. Smith, who considered a bed-profile similar to that of Fig. 2a, showed a single hump only on the free

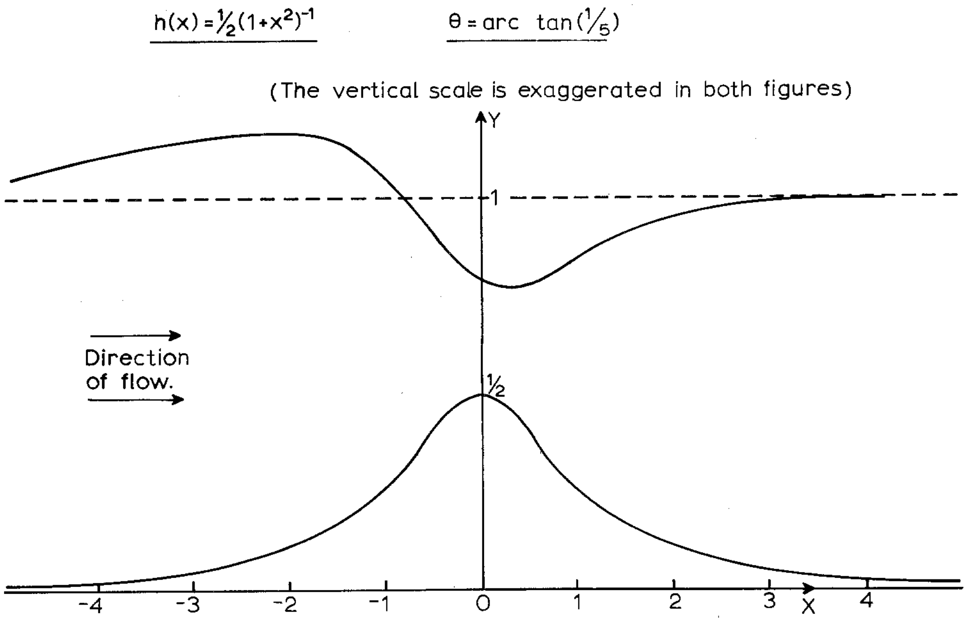

Figure 2a.

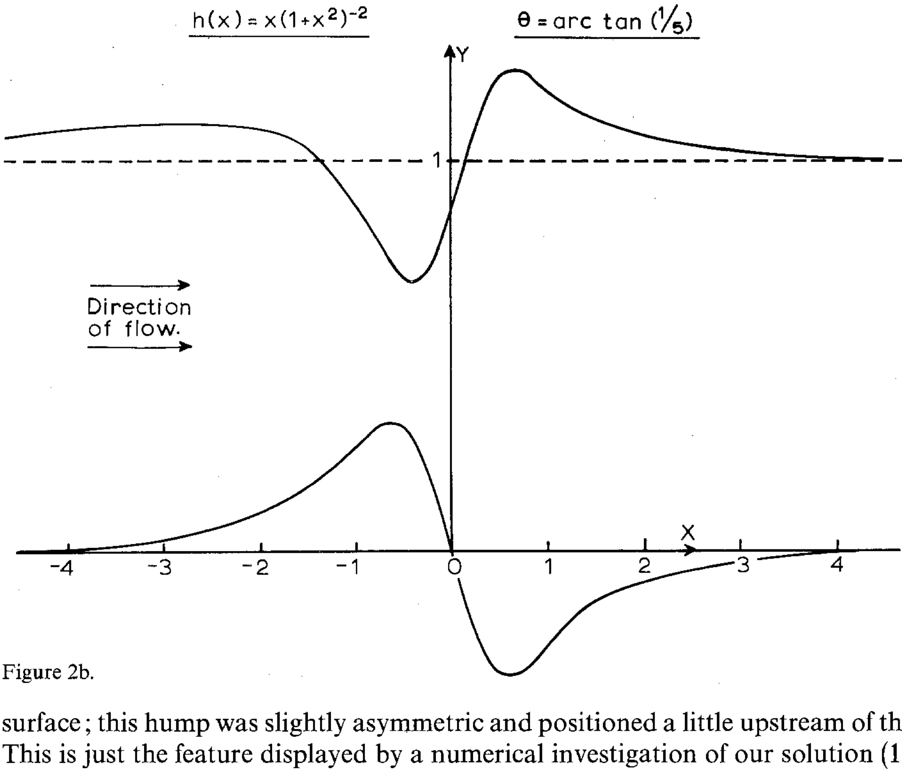

Figure 2b.

surface; this hump was slightly asymmetric and positioned a little upstream of that on the bed. This is just the feature displayed by a numerical investigation of our solution (15). As is often the case, such common ground between different linear and non-linear theories is more typical of the linear than the non-linear extension.

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Appendix

In this appendix the analysis for the derivation of the differential equation (13) is briefly described. To begin, the no-slip conditions (6) on the bed indicate

$$\psi_1 + 2h\psi_2 + 3h^2\psi_3 = O(\varepsilon^4) \tag{A1}$$

and

$$\psi_{0x} + h\psi_{1x} + h^2\psi_{2x} + h^3\psi_{3x} = O(\varepsilon^5)\,. \tag{A2}$$

In the first of these the left hand side in $O(\varepsilon^2)$, while in the other, all but the last term are $O(\varepsilon^3)$. (Although the exact error will always be given in the following, a general rule is that we will keep the first two non-zero terms only in the series expansions with respect to ε.) Next, the kinematic condition on the free surface, (5a), indicates

$$(\psi_1 + 2\eta\psi_2 + 3\eta^2\psi_3)\eta_x + (\psi_{0x} + \eta\psi_{1x} + \eta^2\psi_{2x} + \eta^3\psi_{3x}) = O(\varepsilon^5)\ ; \tag{A3}$$

all except the last term, which is $O(\varepsilon^4)$, are $O(\varepsilon^3)$. Finally the two dynamic conditions (5b,c) show

$$(p_0 + \eta p_1) + 2(\psi_{1x} + 2\eta\psi_{2x}) + (2\psi_2 + 6\eta\psi_3)\eta_x = O(\varepsilon^3)\,, \tag{A4}$$

and

$$(2\psi_2 + 6\eta\psi_3) + (p_0 + \eta p_1)\eta_x = O(\varepsilon^3)\ ; \tag{A5}$$

the first terms in the brackets are $O(\varepsilon)$ in each equation, while the others are $O(\varepsilon^2)$.

The quantities ψ_3 and p_1 can now be found from (12a) and (12c) respectively to the orders of magnitude required. After substitution, the five equations (A1-5) contain the five unknown functions ψ_0, ψ_1, ψ_2, p_0 and η, together with the known function h. It is now a straightforward process to eliminate all the unknown functions except η; the result is the differential equation (13) for $\eta(x)$ in terms of $h(x)$.

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